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# **A New Matrix Formulation of Classical Electrodynamics**

**Part I. Vacuum**

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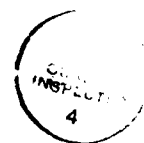
## ABSTRACT

Presented in this paper is a new matrix representation of classical electromagnetic theory. The basis of this representation is a space-time, eight-by-eight differential matrix operator. This matrix operator is initially formulated from the differential form of the Maxwell field equations in vacuum. The resulting matrix formulation of Maxwell's equations allows simple and direct derivation of the electromagnetic wave and charge continuity equations, the Lorentz conditions and definition of the electromagnetic potentials, the Lorentz and Coulomb gauges, the electromagnetic potential wave equations, and Poynting's conservation of energy theorem. A four-dimensional Fourier transform of the matrix equations casts them into an eight-dimensional transfer theorem. The transfer function has an inverse, and this allows the equations to be inverted. This inversion expresses the fields directly in terms of the charge and current source distributions, i.e., without the need for calculating intermediary potentials. This inversion formula is new, for the general scenario of space- and time-dependent sources. A simple pedagogical example is included illustrating use of the formulation.

# CONTENTS

1.0 INTRODUCTION .....	1
2.0 DERIVATION OF THE [M] MATRIX REPRESENTATION .....	2
3.0 MATRIX FORMULATION OF OTHER ELECTROMAGNETIC PHENOMENA .....	9
3.1 ELECTROMAGNETIC FIELD WAVE EQUATIONS AND CHARGE CONTINUITY EQUATIONS .....	9
3.2 ELECTROMAGNETIC POTENTIALS AND LORENTZ CONDITIONS .....	10
3.3 ELECTROMAGNETIC POTENTIAL WAVE EQUATIONS .....	11
3.4 ENERGY CONSERVATION; POYNTING'S THEOREM .....	12
4.0 ELECTROMAGNETIC FIELD FOURIER TRANSFORM REPRESENTATION .....	13
5.0 ELECTROMAGNETIC FIELD POINT SPREAD FUNCTION REPRESENTATION .....	17
6.0 EXAMPLE: INFINITE LINE OF ELECTRIC CHARGE .....	21
7.0 ELECTROMAGNETIC POTENTIAL FOURIER TRANSFORM REPRESENTATION .....	23
8.0 SUMMARY AND CONCLUSIONS .....	24
REFERENCES .....	26
APPENDICES .....	A-1
A: COULOMB GAUGE .....	A-1
B: COVARIANCE OF THE EIGHT-BY-EIGHT MATRIX REPRESENTATION OF MAXWELL'S EQUATIONS .....	B-1
C: INVERSION OF MATRIX [K] .....	C-1

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## 1.0 INTRODUCTION

Present day mathematical descriptions and computations in the field of classical electrodynamics rely heavily on the use of vector calculus. Matrix calculus, on the other hand, finds widespread use in other areas of physics, engineering, and applied mathematics. Structural analysis, vibrational analysis, electrical circuit analysis, and the theory of elasticity are just a few of the disciplines (Pipes, 1963) where matrix calculus has been successfully employed routinely. This paper presents a new formulation of electromagnetic theory from a matrix calculus point of view. We will show the formulation arises from use of a skew-Hermitian space-time eight-by-eight differential matrix operator

$$\begin{bmatrix} -\frac{\partial}{\partial \tau} & 0 & 0 & -\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 \\ 0 & -\frac{\partial}{\partial \tau} & 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & -\frac{\partial}{\partial \tau} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 & -\frac{\partial}{\partial \tau} & 0 & 0 & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 & 0 & -\frac{\partial}{\partial \tau} & 0 & -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial z} \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & -\frac{\partial}{\partial \tau} \end{bmatrix} \equiv [M]. \quad (1)$$

By "skew-Hermitian" we mean  $[M]^H = -[M]$ , where  $[M]^H$  denotes the Hermitian (complex conjugate) transpose of  $[M]$ . The notation

$$\tau \equiv ict, \quad i \equiv \sqrt{-1} \quad (2)$$

is adopted for describing the temporal component of a space-time point. All basic physical quantities of the theory will be four-dimensional, with a complex fourth component as in equation 2.

## 2.0 DERIVATION OF THE [M] MATRIX REPRESENTATION

The fundamental equations of classical electromagnetic phenomena, namely the Maxwell field equations, serve as our starting point. In this section the space-time operator [M], which is of paramount importance to the remainder of the paper, is constructed from the Maxwell field equations. In the Gaussian system of units, the four Maxwell field equations in vector form are given by the following (Jackson, 1962):

**Ampere-Maxwell law**

$$\nabla \times H(r, t) = \left(\frac{1}{c}\right) \frac{\partial}{\partial t} D(r, t) + \left(\frac{4\pi}{c}\right) J^e(r, t) \quad (3a)$$

**Gauss' law for electricity**

$$\nabla \cdot D(r, t) = (4\pi) \rho^e(r, t) \quad (3b)$$

**Faraday's law**

$$\nabla \times E(r, t) = - \left(\frac{1}{c}\right) \frac{\partial}{\partial t} B(r, t) - \left(\frac{4\pi}{c}\right) J^m(r, t) \quad (3c)$$

**Gauss' law for magnetism**

$$\nabla \cdot B(r, t) = (4\pi) \rho^m(r, t) \quad (3d)$$

The order in which the four equations appear above is important in the construction of the space-time operator [M]. In rectangular coordinates

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \quad (4)$$

and

$$r = (x, y, z). \quad (5)$$

The physical quantities appearing in the Maxwell field equations are

$E(r, t)$  – Electric field

$D(r, t)$  – Electric displacement

$B(r, t)$  – Magnetic induction

$H(r, t)$  – Magnetic field

$J^e(r, t)$  – Electric current density

$J^m(r, t)$  – Magnetic current density

$\rho^e(r, t)$  – Electric charge density

$\rho^m(r, t)$  – Magnetic charge density

$\nabla$  – Del operator

$\nabla \bullet$  – Divergence operator

$\nabla \times$  – Curl operator

$(r, t)$  – Space-time point

$c$  – Speed of light in vacuum

Both magnetic charge and current densities (Magid, 1972) have been included in Maxwell's equations for purposes of completeness. These densities, of course, may be set equal to zero since magnetic monopoles have not yet been discovered in nature.

The electric displacement and electric field, as well as the magnetic induction and magnetic field, are related (Jackson, 1962) through the expressions

$$D(r, t) = E(r, t) + (4\pi) P(r, t) \quad (6a)$$

$$B(r, t) = H(r, t) + (4\pi) M(r, t) \quad (6b)$$

where

$P(r, t)$  – Macroscopic polarization

$M(r, t)$  – Macroscopic magnetization.

In this paper, dielectric and magnetic materials are not considered. Hence, we let

$$P(r, t) = 0 \quad (7a)$$

$$M(r, t) = 0. \quad (7b)$$

A subsequent paper will treat dielectric and magnetic materials where both the macroscopic polarization and magnetization vectors are non-zero.

With the use of equations 6 and 7, the four Maxwell equations for vacuum can be rewritten as eight scalar equations. That is,

#### Ampere-Maxwell law

$$\frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y - \left(\frac{1}{c}\right) \frac{\partial}{\partial t} E_x = \left(\frac{4\pi}{c}\right) J_x^e \quad (8a)$$

$$\frac{\partial}{\partial z} H_x - \frac{\partial}{\partial x} H_z - \left(\frac{1}{c}\right) \frac{\partial}{\partial t} E_y = \left(\frac{4\pi}{c}\right) J_y^e \quad (8b)$$

$$\frac{\partial}{\partial x} H_y - \frac{\partial}{\partial y} H_x - \left(\frac{1}{c}\right) \frac{\partial}{\partial t} E_z = \left(\frac{4\pi}{c}\right) J_z^e \quad (8c)$$

#### Gauss' law for electricity

$$\frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = \left(\frac{4\pi}{c}\right) c\rho^e \quad (8d)$$

#### Faraday's law

$$\frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y + \left(\frac{1}{c}\right) \frac{\partial}{\partial t} H_x = -\left(\frac{4\pi}{c}\right) J_x^m \quad (8e)$$

$$\frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z + \left(\frac{1}{c}\right) \frac{\partial}{\partial t} H_y = -\left(\frac{4\pi}{c}\right) J_y^m \quad (8f)$$

$$\frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x + \left(\frac{1}{c}\right) \frac{\partial}{\partial t} H_z = -\left(\frac{4\pi}{c}\right) J_z^m \quad (8g)$$

#### Gauss' law for magnetism

$$\frac{\partial}{\partial x} H_x + \frac{\partial}{\partial y} H_y + \frac{\partial}{\partial z} H_z = \left(\frac{4\pi}{c}\right) c\rho^m. \quad (8h)$$



The eight scalar equations can now be combined into a single matrix equation involving an eight-by-six differential matrix operator:

$$\begin{bmatrix}
 -\frac{\partial}{\partial \tau} & 0 & 0 & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
 0 & -\frac{\partial}{\partial \tau} & 0 & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\
 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
 \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & 0 & 0 \\
 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & -\frac{\partial}{\partial \tau} & 0 & 0 \\
 \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial \tau} & 0 \\
 -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & -\frac{\partial}{\partial \tau} \\
 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
 \end{bmatrix}
 \begin{bmatrix}
 iE_x \\
 iE_y \\
 iE_z \\
 H_x \\
 H_y \\
 H_z
 \end{bmatrix}
 = \left(\frac{4\pi}{c}\right)
 \begin{bmatrix}
 J_x^e \\
 J_y^e \\
 J_z^e \\
 ic\rho^e \\
 -iJ_x^m \\
 -iJ_y^m \\
 -iJ_z^m \\
 c\rho^m
 \end{bmatrix}
 \quad (9)$$

Other authors have formed electromagnetic matrix operators similar to equation 9, but either under less general conditions, or with different goals in mind (Mayes, 1967; Eisele and Mason, 1970). Mayes (1967) assumes the special case of time-harmonic fields due to radiating antennas, so that time is suppressed in the formulation. The effect is suppression of Maxwell's equations 3b and 3d, resulting in a six-by-six matrix operator formulation instead of eight-by-six matrix operator formulation (equation 9). Eisele and Mason (1970) assume zero source strengths, and form complex field quantities,  $E_x + iE_y, E_z, H_x + iH_y, H_z$ , with the goal of showing Maxwell's equations reduce to a Dirac-like equation for a four-component electromagnetic spinor field with zero mass.

Also, no other authors, as far as we know, take the step described next of broadening the long (eight-by-six) matrix in equation 9 into an associated square (eight-by-eight) matrix, with attendant theoretical benefits.

We are now in a position to construct the desired eight-by-eight space-time operator [M]. The key to this construction is to first replace the eight-by-six matrix operator appearing in equation 9 by an eight-by-eight matrix operator containing two additional

columns. That is,

$$\begin{bmatrix}
 -\frac{\partial}{\partial \tau} & 0 & 0 & \hat{m}_{14} & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \hat{m}_{18} \\
 0 & -\frac{\partial}{\partial \tau} & 0 & \hat{m}_{24} & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & \hat{m}_{28} \\
 0 & 0 & -\frac{\partial}{\partial \tau} & \hat{m}_{34} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & \hat{m}_{38} \\
 \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \hat{m}_{44} & 0 & 0 & 0 & \hat{m}_{48} \\
 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \hat{m}_{54} & -\frac{\partial}{\partial \tau} & 0 & 0 & \hat{m}_{58} \\
 \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & \hat{m}_{64} & 0 & -\frac{\partial}{\partial \tau} & 0 & \hat{m}_{68} \\
 -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & \hat{m}_{74} & 0 & 0 & -\frac{\partial}{\partial \tau} & \hat{m}_{78} \\
 0 & 0 & 0 & \hat{m}_{84} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \hat{m}_{88}
 \end{bmatrix}
 \begin{bmatrix}
 iE_x \\
 iE_y \\
 iE_z \\
 0 \\
 H_x \\
 H_y \\
 H_z \\
 0
 \end{bmatrix}
 = \left(\frac{4\pi}{c}\right)
 \begin{bmatrix}
 j_x^e \\
 j_y^e \\
 j_z^e \\
 ic\rho^e \\
 -ij_x^m \\
 -ij_y^m \\
 -ij_z^m \\
 c\rho^m
 \end{bmatrix}. \quad (10)$$

Columns 4 and 8 are the newly inserted columns. Their elements  $\hat{m}_{ij}$  are as yet undefined. To null-out the effects of the two new columns, additional zeros must be inserted in the electric and magnetic field vector at positions 4 and 8. This eight-by-eight matrix operator representation of Maxwell's equations is now entirely equivalent to the original vector representation, regardless of choice of the added elements  $\hat{m}_{ij}$ . We next choose useful, specific values for the added  $\hat{m}_{ij}$ .

The motivation for choice of the added elements  $\hat{m}_{ij}$  is as follows. The off-diagonal known elements in the eight-by-eight matrix have an *antisymmetric* arrangement. For example, element  $\hat{m}_{15}$  is  $\frac{\partial}{\partial z}$  and  $\hat{m}_{52}$  is  $-\frac{\partial}{\partial z}$ . This arrangement suggests that we choose the added elements  $\hat{m}_{ij}$  such that antisymmetry is preserved. This preservation requires that we choose the off diagonal elements  $\hat{m}_{ij}$  as follows:

$$\hat{m}_{14} = \hat{m}_{58} = -\frac{\partial}{\partial x} \quad (11a)$$

$$\hat{m}_{24} = \hat{m}_{68} = -\frac{\partial}{\partial y} \quad (11b)$$

$$\hat{m}_{34} = \hat{m}_{78} = -\frac{\partial}{\partial z} \quad (11c)$$

and

$$\hat{m}_{54} = \hat{m}_{64} = \hat{m}_{74} = \hat{m}_{84} = 0 \quad (12a)$$

$$\hat{m}_{18} = \hat{m}_{28} = \hat{m}_{38} = \hat{m}_{48} = 0. \quad (12b)$$

(Actually, assigning  $\hat{m}_{84} = \hat{m}_{48} = 0$  is arbitrary; these elements can be assigned as  $\hat{m}_{84} = b$  and  $\hat{m}_{48} = -b$ , with  $b$  arbitrary. We choose  $b = 0$  for simplicity.)

Next, the diagonal elements  $\hat{m}_{44}$  and  $\hat{m}_{88}$  have to be specified. If these elements are chosen according to the equation

$$\hat{m}_{44} = \hat{m}_{88} = -\frac{\partial}{\partial \tau} \quad (13)$$

then the conditions for the Lorentz gauge are satisfied. This will be seen in the next section. If, however, these elements are chosen according to the equation

$$\hat{m}_{44} = \hat{m}_{88} = 0 \quad (14)$$

then the conditions of the Coulomb gauge are satisfied. See appendix A for more details on the Coulomb gauge. Substituting the Lorentz-gauge choice, equation 13, and the antisymmetric choices, equations 11 and 12, in the general form, equation 10, leads to the following eight-by-eight matrix representation of Maxwell's equations:

$$\begin{bmatrix} -\frac{\partial}{\partial \tau} & 0 & 0 & -\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 \\ 0 & -\frac{\partial}{\partial \tau} & 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & -\frac{\partial}{\partial \tau} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 & -\frac{\partial}{\partial \tau} & 0 & 0 & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 & 0 & -\frac{\partial}{\partial \tau} & 0 & -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial z} \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & -\frac{\partial}{\partial \tau} \end{bmatrix} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} J_x^e \\ J_y^e \\ J_z^e \\ ic\rho^e \\ -ij_x^m \\ -ij_y^m \\ -ij_z^m \\ c\rho^m \end{bmatrix} \quad (15)$$

The left-hand matrix is the matrix in equation 1 we previously called  $[M]$ . Derivation of this matrix form (equation 15) of the Maxwell field equations is the chief result of this section. This matrix will be used throughout the remainder of the paper. This matrix permits simple matrix operations upon equation 15 to derive the major phenomena of electromagnetic theory. This derivation is the subject of section 3. It also permits direct inversion for the fields  $E$  and  $H$  in terms of the sources, without the need for calculating intermediary potentials. This inversion is the subject of sections 4 and 5.

Equation 15 can be rewritten in a more compact form

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \quad \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \equiv [M], \quad (16)$$

in terms of four-by-four submatrices

$$[M_1] \equiv \begin{bmatrix} -\frac{\partial}{\partial \tau} & 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & -\frac{\partial}{\partial \tau} & 0 & -\frac{\partial}{\partial y} \\ 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & -\frac{\partial}{\partial \tau} \end{bmatrix} \quad \text{and} \quad [M_2] \equiv \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (17)$$

four-by-one vector fields

$$[f_1] \equiv \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \end{bmatrix} \quad \text{and} \quad [f_2] \equiv \begin{bmatrix} H_x \\ H_y \\ H_z \\ 0 \end{bmatrix}, \quad (18)$$

and relativistic four-vector sources

$$[s_1] \equiv \begin{bmatrix} J_x^e \\ J_y^e \\ J_z^e \\ ic\rho^e \end{bmatrix} \quad \text{and} \quad [s_2] \equiv \begin{bmatrix} -iJ_x^m \\ -iJ_y^m \\ -iJ_z^m \\ c\rho^m \end{bmatrix}. \quad (19)$$

In summary, the matrix equation 16 is equivalent to the Maxwell field equations in vacuum. Appendix B shows the matrix representation (equation 16) is in covariant form. This development will continue in section 4 where four-dimensional Fourier transforms will be taken in order to achieve inversion of the matrix field equations.

### 3.0 MATRIX FORMULATION OF OTHER ELECTROMAGNETIC PHENOMENA

The matrix representation (equation 16) of the Maxwell field equations will imply, in simple fashion, some milestone effects of electromagnetic theory. Simple matrix operations upon the space-time operator [M] constructed will be shown to derive (a) the electromagnetic wave and charge continuity equations, (b) the Lorentz conditions and definition of the electromagnetic potentials, (c) the electromagnetic potential wave equations, and (d) Poynting's conservation of energy theorem.

#### 3.1 ELECTROMAGNETIC FIELD WAVE EQUATIONS AND CHARGE CONTINUITY EQUATIONS

Multiply both sides of matrix equation 16 by the complex conjugate of the space-time operator [M]. This gives

$$\begin{bmatrix} D & O \\ O & D \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad (20)$$

where

$$\begin{bmatrix} D \end{bmatrix} \equiv \begin{bmatrix} \square^2 & 0 & 0 & 0 \\ 0 & \square^2 & 0 & 0 \\ 0 & 0 & \square^2 & 0 \\ 0 & 0 & 0 & \square^2 \end{bmatrix}, \quad \begin{bmatrix} O \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (21)$$

and  $\square^2$  is the D'Alembertian operator (Feynman, Leighton, and Sands, 1964) defined by

$$\square^2 \equiv -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \quad (22)$$

The matrix equation 20 is equivalent to eight scalar equations. These scalar equations are equivalent to the vector electromagnetic field wave equations

$$\nabla^2 E - \left(\frac{1}{c^2}\right) \frac{\partial^2 E}{\partial t^2} = (4\pi) \nabla \rho^e + \left(\frac{4\pi}{c^2}\right) \frac{\partial}{\partial t} J^e + \left(\frac{4\pi}{c}\right) \nabla \times J^m \quad (23a)$$

$$\nabla^2 H - \left(\frac{1}{c^2}\right) \frac{\partial^2 H}{\partial t^2} = (4\pi) \nabla \rho^m + \left(\frac{4\pi}{c^2}\right) \frac{\partial}{\partial t} J^m - \left(\frac{4\pi}{c}\right) \nabla \times J^e \quad (23b)$$

and the electric and magnetic charge continuity equations

$$\nabla \cdot J^e + \frac{\partial}{\partial t} \rho^e = 0 \quad (24a)$$

$$\nabla \cdot J^m + \frac{\partial}{\partial t} \rho^m = 0. \quad (24b)$$

In summary, a single matrix multiplication of both sides of the Maxwell field equations (equation 16) by the complex conjugate of the space-time operator [M] yields a single matrix equation which is equivalent to both the electromagnetic field wave equations and the charge continuity equations.

### 3.2 ELECTROMAGNETIC POTENTIALS AND LORENTZ CONDITIONS

We found before that the space-time operator [M] defines the eight scalar Maxwell field equations. We now observe that the complex conjugate of [M] also provides the definition of the electromagnetic fields in terms of the familiar vector and scalar potentials. In particular, for the Lorentz gauge, the matrix equation

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad (25)$$

where

$$\begin{bmatrix} p_1 \end{bmatrix} \equiv \begin{bmatrix} A_x^e \\ A_y^e \\ A_z^e \\ i\phi^e \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_2 \end{bmatrix} \equiv \begin{bmatrix} -iA_x^m \\ -iA_y^m \\ -iA_z^m \\ \phi^m \end{bmatrix} \quad (26)$$

is equivalent to eight scalar equations. Six of these are equivalent to the two vector equations which define the relationship between the electromagnetic potentials and the electromagnetic fields

$$E = -\nabla \phi^e - \left(\frac{1}{c}\right) \frac{\partial}{\partial t} A^e - \nabla \times A^m \quad (27a)$$

$$H = -\nabla \phi^m - \left(\frac{1}{c}\right) \frac{\partial}{\partial t} A^m + \nabla \times A^e. \quad (27b)$$

The remaining two scalar equations correspond to the Lorentz conditions

$$\nabla \cdot A^e + \left(\frac{1}{c}\right) \frac{\partial}{\partial t} \phi^e = 0 \quad (28a)$$

$$\nabla \cdot A^m + \left(\frac{1}{c}\right) \frac{\partial}{\partial t} \phi^m = 0. \quad (28b)$$

Results, equations 27 and 28, are easily verified by explicit multiplication of the right-hand side of matrix equation 25.

It is interesting that the two vectors  $[p_1], [p_2]$  are, by their definitions (equation 26), relativistic four-vectors. A comparison of matrix equations 16 and 25 discloses the curious fact that the same operator  $[M]$  that operates upon the fields  $f_i, i=1,2$  to give the sources  $s_i$  also operates upon the potentials  $p_i$  to give the fields. There is an interesting symmetry between the two effects.

### 3.3 ELECTROMAGNETIC POTENTIAL WAVE EQUATIONS

We know the electromagnetic vector and scalar potentials satisfy inhomogeneous wave equations. Once again, we show this by simple matrix multiplication. First substitute the matrix expression (equation 25) into the matrix representation (equation 16) of Maxwell's equations. This gives

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \quad (29)$$

However, for the Lorentz-gauge choice of  $[M]$ ,

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} = \begin{bmatrix} D & O \\ O & D \end{bmatrix} \quad (30)$$

by direct evaluation of the matrix product. Then, by equations 29 and 30, the following matrix representation of the electromagnetic potential wave equations is obtained

$$\begin{bmatrix} D & O \\ O & D \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \quad (31)$$

By explicit evaluation of the left-hand matrix product, this single matrix equation is equivalent to eight scalar equations. Six of these scalar equations are equivalent to the following pair of inhomogeneous wave equations for the electric and magnetic vector potentials

$$\nabla^2 A^e - \left(\frac{1}{c^2}\right) \frac{\partial^2 A^e}{\partial t^2} = -\left(\frac{4\pi}{c}\right) J^e \quad (32a)$$

$$\nabla^2 A^m - \left(\frac{1}{c^2}\right) \frac{\partial^2 A^m}{\partial t^2} = -\left(\frac{4\pi}{c}\right) J^m. \quad (32b)$$

The remaining two represent the inhomogeneous wave equations for the electric and magnetic scalar potentials

$$\nabla^2 \phi^e - \left(\frac{1}{c^2}\right) \frac{\partial^2 \phi^e}{\partial t^2} = -(4\pi) \rho^e \quad (33a)$$

$$\nabla^2 \phi^m - \left(\frac{1}{c^2}\right) \frac{\partial^2 \phi^m}{\partial t^2} = -(4\pi) \rho^m. \quad (33b)$$

### 3.4 ENERGY CONSERVATION; POYNTING'S THEOREM

The law of conservation of energy, often called Poynting's theorem (Jackson, 1962), is an important milestone of electromagnetic theory. Again simple matrix manipulation of equation 16 will accomplish the derivation. Multiply both sides of equation 16 by the Hermitian conjugate of the electromagnetic field vector,  $\begin{bmatrix} f_1 & f_2 \end{bmatrix}^*$ . Directly,

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix}^* \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} f_1 & f_2 \end{bmatrix}^* \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \quad (34)$$



This is a single scalar equation representing Poynting's theorem

$$\nabla \cdot \mathbf{S} + \frac{\partial}{\partial t} u + \mathbf{E} \cdot \mathbf{J}^e + \mathbf{H} \cdot \mathbf{J}^m = 0 \quad (35)$$

if we define quantity  $u$  to obey

$$u \equiv \left( \frac{1}{8\pi} \right) ( \mathbf{E} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{H} ), \quad (36)$$

where  $u$  is the total energy density of the electromagnetic fields, and define  $\mathbf{S}$  by

$$\mathbf{S} \equiv \left( \frac{c}{4\pi} \right) ( \mathbf{E} \times \mathbf{H} ), \quad (37)$$

where  $\mathbf{S}$  is the Poynting vector representing energy flow. Again, this is easily verified by explicit multiplication out of the matrix products in equation 34.

## 4.0 ELECTROMAGNETIC FIELD FOURIER TRANSFORM REPRESENTATION

This section gives a method for inverting the Maxwell field equations. The Maxwell field equations have been written in a compact matrix form involving the square eight-by-eight space-time differential operator [M]. With the use of four-dimensional Fourier transform theory, the matrix form of Maxwell's equations can be easily cast from the differential form into an algebraic form. The algebraic form is an eight-dimensional transfer theorem in  $K$ -space, linearly connecting the Fourier transform of the fields to those of the sources. The connection is through an eight-dimensional transfer function, whose inverse is known. This, in turn, permits inversion of the field equations. The fields are then directly known in terms of the charge and current source distributions. (By "direct" we mean without the need for calculating intermediary potential functions.) Alternative computation of the electromagnetic potentials from the charge and current source distributions is presented in section 7.

A three dimensional Fourier transform approach to direct inversion was championed by Mayes (1985). This inversion was for the special case of time-harmonic fields due to antennas. Also, inversion was made by recourse to Green's functions (i.e., use of vector calculus) rather than by direct use of matrix transfer theorem (equation 45).

We first define the four-dimensional Fourier transform pair

$$\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} e^{-i(K \cdot R)} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} dR \quad (38a)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} e^{+i(K \cdot R)} \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} dK \quad (38b)$$

where

$$R = (x, y, z, \tau) = (r, \tau) \quad \text{and} \quad K = (k_x, k_y, k_z, k_\tau) = (\mathbf{k}, k_\tau) \quad (39a)$$

$$dR = dx \, dy \, dz \, d\tau \quad \text{and} \quad dK = dk_x \, dk_y \, dk_z \, d\omega \quad (39b)$$

$$\tau = ict \quad \text{and} \quad k_\tau = i\omega/c. \quad (39c)$$

Next, multiply both sides of equation 38b by the space-time operator [M]

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} e^{+i(K \cdot R)} \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} dK. \quad (40)$$

Operator [M] was absorbed under the integral sign since the integration is with respect to  $K$ , while [M] contains space- and time-derivatives, independent of  $K$ . We can easily show that

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} e^{+i(K \cdot R)} = \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} e^{+i(K \cdot R)} \quad (41)$$

where

$$\begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} = i \begin{bmatrix} -k_\tau & 0 & 0 & -k_x & 0 & -k_z & k_y & 0 \\ 0 & -k_\tau & 0 & -k_y & k_z & 0 & -k_x & 0 \\ 0 & 0 & -k_\tau & -k_z & -k_y & k_x & 0 & 0 \\ k_x & k_y & k_z & -k_\tau & 0 & 0 & 0 & 0 \\ 0 & -k_z & k_y & 0 & -k_\tau & 0 & 0 & -k_x \\ k_z & 0 & -k_x & 0 & 0 & -k_\tau & 0 & -k_y \\ -k_y & k_x & 0 & 0 & 0 & 0 & -k_\tau & -k_z \\ 0 & 0 & 0 & 0 & k_x & k_y & k_z & -k_\tau \end{bmatrix} \equiv i[K]. \quad (42)$$

Then by equation 40,

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} e^{+i(K \cdot R)} dK. \quad (43)$$

Next we denote the Fourier transform of  $[s_1 s_2]^T$  by  $[\tilde{s}_1 \tilde{s}_2]^T$ , where  $T$  is the transpose. Equation 16 becomes

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left( \frac{1}{\pi c} \right) \iiint \int_{-\infty}^{+\infty} e^{+i(K \cdot R)} \begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix} dK. \quad (44)$$

Setting equations 43 and 44 equal to one another gives the following Fourier transform representation of Maxwell's equations

$$\begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} = \left( \frac{4\pi}{c} \right) \begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix}. \quad (45)$$

This is an electromagnetic transfer theorem, the direct counterpart to well-known transfer theorems in optics (Goodman, 1968) circuit theory, etc. The left-hand side matrix  $[K]$  is an eight-by-eight transfer function. It has an inverse, as discussed next.

Transfer function matrix  $[K]$  has elements that are not derivative operators. The elements of  $[K]$  are purely algebraic, i.e. pure numbers in specific cases. Therefore,  $[K]$  can potentially have an inverse in contrast to matrix  $[M]$  in direct-space representation

(equation 16), which consists of differential operators and hence cannot have an inverse in the ordinary sense. The very reason for taking a Fourier approach was, in fact, to achieve a representation with a realizable inverse. The inverse of  $[K]$  is found to exist, and is given by (see appendix C)

$$\begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^{-1} = \left(-\frac{1}{K^2}\right) \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* \quad (46)$$

where

$$K^2 = k_x^2 + k_y^2 + k_z^2 - (\omega^2/c^2). \quad (47)$$

This is easily verified by explicitly evaluating the matrix product of the right-hand side of equation 46 with  $[K]$ . The result is a diagonal identity matrix, as required.

With  $[K]^{-1}$  known, equation 45 may be inverted for the fields in Fourier-space

$$\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix} = \left(-\frac{4\pi}{cK^2}\right) \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* \begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix} \quad (48)$$

by identity (equation 46). Finally, substituting equation 48 into equation 38b gives the fields *per se*,

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\frac{1}{c}\right) \iiint \int_{-\infty}^{+\infty} \left(\frac{-1}{\pi K^2}\right) \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* \begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix} e^{i(\mathbf{K} \cdot \mathbf{R})} d\mathbf{K}. \quad (49)$$

The poles at  $K = 0$  are avoided by recourse to Cauchy principal value integration (Jackson, 1962) in analytic problems; or by simply avoiding the points  $K = 0$  during *numerical* integration of equation 49 in nonanalytic problems.

Briefly consider experimental cases where the sources  $[\tilde{s}_1], [\tilde{s}_2]$  are known with inevitable error, e.g., due to random noise. In this case equation 45 becomes a typical ill-posed, inverse source problem (Ferwerda, 1978). Inversion formula (equation 49) would then have to be modified so as to avoid unduly magnifying the noise at points  $K \rightarrow 0$ . Appropriate window functions that smoothly go to zero as  $K \rightarrow 0$  could be used, e.g., to modulate the integrand of equation 49 for this purpose. Or, if statistical knowledge of power spectra  $\langle |\tilde{s}_i(K)|^2 \rangle, i = 1, 2$  is forthcoming, this power spectra could be used to form a Wiener filter-based inverse solution.

We have thereby achieved our goal of solving for the electromagnetic fields directly in terms of the sources. In fact, equation 49 shows that the *four-dimensional Fourier transforms of the sources* must be known as inputs. In cases where these inputs cannot be computed analytically, numerical use of the four-dimensional Fast Fourier transform (4D-FFT) can be made. Interestingly, rectangular coordinates can then always be used, regardless of the geometrical shapes of the sources.

The Fourier-based approach (equation 49) to inversion appears to be a new one in the general scenario of space- and time-dependent sources. Interestingly, in preceding sections, the approach consists of four-dimensional quantities, and hence continues a covariant development.

## 5.0 ELECTROMAGNETIC FIELD POINT SPREAD FUNCTION REPRESENTATION

Inverse-solution (equation 49) is essentially the Fourier transform of the product of two functions. By the convolution theorem (Bracewell, 1965) this equals the convolution of the Fourier transforms of the two functions. The convolution is then in direct-space (*and not K - space*), and hence is in terms of source functions  $s_1$  and  $s_2$  directly. We develop such a direct-space solution next.

We note the following Fourier transform pair relations

$$G(R) = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} \left( \frac{1}{\pi K^2} \right) e^{+i(K \cdot R)} dK \quad (50a)$$

$$\left( \frac{1}{\pi K^2} \right) = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} G(R) e^{-i(K \cdot R)} dR \quad (50b)$$

for a Green's function (Jackson, 1962) or point spread function given by

$$G(R) = \frac{\delta\left(\frac{|r|}{c} - t\right)}{|r|} \quad (51)$$

The greek letter  $\delta$  denotes Dirac delta function. Next we form the following partial derivatives of the Green's function from equation 50a,

$$\frac{\partial}{\partial u} G(R) = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} \left( \frac{i k_u}{\pi K^2} \right) e^{i(K \cdot R)} dK. \quad (52a)$$

for  $u = x, y, z, \text{ or } \tau$

Equation 52a shows that  $\frac{\partial}{\partial u} G(R)$  has as its Fourier transform mate the quantity

$$\left( \frac{i k_u}{\pi K^2} \right) = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} \frac{\partial}{\partial u} G(R) e^{-i(K \cdot R)} dR. \quad (52b)$$

for  $u = x, y, z, \text{ or } \tau$

Note, quantity  $k_u$  in identity (equation 52b) represents the general element of matrix [K] given by equation 42. With repeated use of identity (equation 52b) for each element of [K] we form the following matrix Fourier transform pair

$$\left( \frac{-1}{\pi K^2} \right) \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix} e^{-i(K \cdot R)} dR \quad (53a)$$

$$\begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix} = \left( \frac{1}{4\pi^2} \right) \iiint \int_{-\infty}^{+\infty} \left( \frac{-1}{\pi K^2} \right) \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* e^{+i(K \cdot R)} dK \quad (53b)$$

where

$$\begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix} \equiv [N] \equiv \begin{bmatrix} Q_\tau & 0 & 0 & -Q_x & 0 & -Q_z & Q_y & 0 \\ 0 & Q_\tau & 0 & -Q_y & Q_z & 0 & -Q_x & 0 \\ 0 & 0 & Q_\tau & -Q_z & -Q_y & Q_x & 0 & 0 \\ Q_x & Q_y & Q_z & Q_\tau & 0 & 0 & 0 & 0 \\ 0 & -Q_z & Q_y & 0 & Q_\tau & 0 & 0 & -Q_x \\ Q_z & 0 & -Q_x & 0 & 0 & Q_\tau & 0 & -Q_y \\ -Q_y & Q_x & 0 & 0 & 0 & 0 & Q_\tau & -Q_z \\ 0 & 0 & 0 & 0 & Q_x & Q_y & Q_z & Q_\tau \end{bmatrix} \quad (54)$$

with elements

$$Q_u(R) \equiv \frac{\partial}{\partial u} G(R). \quad (55)$$

The partial derivatives of the Green's function are given by

$$\frac{\partial}{\partial x} G(R) = - \left( \frac{x}{|r|^3} \right) \delta \left( \frac{|r|}{c} - t \right) + \left( \frac{x}{c|r|^2} \right) \delta' \left( \frac{|r|}{c} - t \right) \quad (56a)$$

$$\frac{\partial}{\partial y} G(R) = - \left( \frac{y}{|r|^3} \right) \delta \left( \frac{|r|}{c} - t \right) + \left( \frac{y}{c|r|^2} \right) \delta' \left( \frac{|r|}{c} - t \right) \quad (56b)$$

$$\frac{\partial}{\partial z} G(R) = - \left( \frac{z}{|r|^3} \right) \delta \left( \frac{|r|}{c} - t \right) + \left( \frac{z}{c|r|^2} \right) \delta' \left( \frac{|r|}{c} - t \right) \quad (56c)$$

$$\frac{\partial}{\partial \tau} G(R) = \left( \frac{i}{c|r|} \right) \delta' \left( \frac{|r|}{c} - t \right). \quad (56d)$$

According to plan, we now apply the convolution theorem to inversion formula (equation 49) which becomes the convolution of  $[s_1 s_2]^T$  with the Fourier transform of  $-[K] / (\pi K^2)$ . By identity (equation 53a) the latter is  $[N]$ . Hence the inversion formula becomes

$$\begin{bmatrix} f_1(R) \\ f_2(R) \end{bmatrix} = \left( \frac{1}{c} \right) \iiint \int_{-\infty}^{+\infty} \begin{bmatrix} N_1(R-R') & N_2(R-R') \\ N_2(R-R') & N_1(R-R') \end{bmatrix} \begin{bmatrix} s_1(R') \\ s_2(R') \end{bmatrix} dR'. \quad (57)$$

Integration of equation 57 over time yields the desired inversion formula in direct-space. The fields are the results of three added convolution effects

$$\begin{bmatrix} f_1(r, t) \\ f_2(r, t) \end{bmatrix} = \begin{bmatrix} f_1(r, t) \\ f_2(r, t) \end{bmatrix}_1 + \begin{bmatrix} f_1(r, t) \\ f_2(r, t) \end{bmatrix}_2 + \begin{bmatrix} f_1(r, t) \\ f_2(r, t) \end{bmatrix}_3, \quad (58)$$

where the individual convolutions are

$$\begin{bmatrix} f_1(r, t) \\ f_2(r, t) \end{bmatrix}_1 = \iiint_{-\infty}^{+\infty} \begin{bmatrix} X_1(r-r') & X_2(r-r') \\ X_2(r-r') & X_1(r-r') \end{bmatrix} \begin{bmatrix} s_1(r', t') \\ s_2(r', t') \end{bmatrix}_{ret} dr', \quad (59a)$$

$$\begin{bmatrix} f_1(r, t) \\ f_2(r, t) \end{bmatrix}_2 = \iiint_{-\infty}^{+\infty} \begin{bmatrix} Y_1(r-r') & Y_2(r-r') \\ Y_2(r-r') & Y_1(r-r') \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t'} s_1(r', t') \\ \frac{\partial}{\partial t'} s_2(r', t') \end{bmatrix}_{ret} dr', \quad (59b)$$

and

$$\begin{bmatrix} f_1(r, t) \\ f_2(r, t) \end{bmatrix}_3 = \iiint_{-\infty}^{+\infty} \begin{bmatrix} Z_1(r-r') & Z_2(r-r') \\ Z_2(r-r') & Z_1(r-r') \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t'} s_1(r', t') \\ \frac{\partial}{\partial t'} s_2(r', t') \end{bmatrix}_{ret} dr'. \quad (59c)$$

Quantities  $X_i, Y_i, Z_i, i = 1, 2$  are defined by

$$\begin{bmatrix} X_1(r) & X_2(r) \\ X_2(r) & X_1(r) \end{bmatrix} = \frac{1}{c|r|^3} \begin{bmatrix} J & H \\ H & J \end{bmatrix}, \quad (60a)$$

$$\begin{bmatrix} Y_1(r) & Y_2(r) \\ Y_2(r) & Y_1(r) \end{bmatrix} = \frac{1}{c^2|r|^2} \begin{bmatrix} J & H \\ H & J \end{bmatrix}, \quad (60b)$$

$$\begin{bmatrix} Z_1(r) & Z_2(r) \\ Z_2(r) & Z_1(r) \end{bmatrix} = -\frac{i}{c^2|r|} \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \quad (60c)$$



where

$$\begin{bmatrix} J \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & z \\ -x & -y & -z & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} H \end{bmatrix} = \begin{bmatrix} 0 & z & -y & 0 \\ -z & 0 & x & 0 \\ y & -x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (61a)$$

$$\begin{bmatrix} I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} O \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (61b)$$

The brackets  $[\ ]_{ret}$  appearing in equation 59 means that the time  $t'$  is to be evaluated at the retarded time

$$t' = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}. \quad (62)$$

Inversion formula (equations 58 and 59) for the fields appears to be new. We emphasize the inversion formula calculates the fields directly from the sources, with no need to compute potential functions first. From the form of equations 59a, b, and c, quantities  $X_i, Y_i, Z_i$  play the role of effective Green's functions, or point spread functions, for the inversion process.

## 6.0 EXAMPLE: INFINITE LINE OF ELECTRIC CHARGE

Merely to exemplify use of the inversion approach (equations 58 through 61), we apply the approach to the following simple problem. An infinite line of electric charge is uniformly distributed along the z-axis, with a time-independent electric charge density function given by

$$\rho^e(\mathbf{r}', t') = \rho^e(\mathbf{r}') = \lambda \delta(x') \delta(y'). \quad (63)$$

The constant,  $\lambda$ , represents the electric charge per unit length. Since the charge-distribution is time-independent, then it is only necessary to use the first convolution integral in equation 58 in calculating the electromagnetic fields,

$$\begin{bmatrix} f_1(\mathbf{r}, t) \\ f_2(\mathbf{r}, t) \end{bmatrix} = \iiint_{-\infty}^{+\infty} \begin{bmatrix} X_1(\mathbf{r} - \mathbf{r}') & X_2(\mathbf{r} - \mathbf{r}') \\ X_2(\mathbf{r} - \mathbf{r}') & X_1(\mathbf{r} - \mathbf{r}') \end{bmatrix} \begin{bmatrix} s_1(\mathbf{r}', t') \\ s_2(\mathbf{r}', t') \end{bmatrix}_{ret} d\mathbf{r}'. \quad (64)$$

The charge and current density vector in equation 64 for this example is given by

$$\begin{bmatrix} s_1(r', t') \\ s_2(r', t') \end{bmatrix}_{ret} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ic\rho^e(r') \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (65)$$

We can easily show using equations 65, 60a, and 61a that

$$\begin{bmatrix} X_1(r-r') & X_2(r-r') \\ X_2(r-r') & X_1(r-r') \end{bmatrix} \begin{bmatrix} s_1(r', t') \\ s_2(r', t') \end{bmatrix}_{ret} = \frac{i\lambda\delta(x')\delta(y')}{|r-r'|^3} \begin{bmatrix} + (x-x') \\ + (y-y') \\ + (z-z') \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (66)$$

Substitution of equation 66 back into equation 64 leads to the result

$$\begin{bmatrix} iE_x(r, t) \\ iE_y(r, t) \\ iE_z(r, t) \\ 0 \\ H_x(r, t) \\ H_y(r, t) \\ H_z(r, t) \\ 0 \end{bmatrix} = \left( \frac{2\lambda i}{x^2 + y^2} \right) \begin{bmatrix} +x \\ +y \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (67)$$

which is equivalent to the well known results

$$E_x(r, t) = \frac{+2\lambda x}{[x^2 + y^2]} \quad E_y(r, t) = \frac{+2\lambda y}{[x^2 + y^2]} \quad E_z(r, t) = 0 \quad (68a)$$

$$H_x(r, t) = 0 \quad H_y(r, t) = 0 \quad H_z(r, t) = 0. \quad (68b)$$

## 7.0 ELECTROMAGNETIC POTENTIAL FOURIER TRANSFORM REPRESENTATION

We showed, in sections 4 and 5, a solution for the fields directly in terms of the sources. Using the same general approach, it is possible to alternatively express the potential functions in terms of the sources. Again, simple matrix manipulations and four-dimensional Fourier transforms are used. For brevity, we only give the results.

Define  $[\tilde{p}_1 \ \tilde{p}_2]^T$  to be the Fourier transform (as in equation 38a) of the potentials  $[p_1 \ p_2]^T$ . Then the spectra  $[\tilde{s}_1 \ \tilde{s}_2]^T$  of the sources connect with the potentials through

$$\begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \left(\frac{1}{K^2}\right) \begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix}, \quad K^2 \equiv k_x^2 + k_y^2 + k_z^2 - (\omega^2/c^2). \quad (69)$$

The derivation is analogous to equations 40 through 48. This is a transfer theory analogous to optical-transfer theory (Goodman, 1968). What might be called an "electromagnetic transfer function" is quantity  $1/K^2$ . As an inversion formula, equation 69 would be convenient to digitally implement by the use of a four-dimensional fast Fourier transform algorithm.

Solution (equation 69) for the potentials corresponds to solution (equation 48) for the fields. Both are transfer theorems, and both can be ill-posed as  $K$  approaches zero. A way out is to cast the solutions in direct-space. In the field calculations case, this gave rise to solution equations 58 and 59. In the potential calculation case, the corresponding solution is

$$\begin{bmatrix} p_1(r, t) \\ p_2(r, t) \end{bmatrix} = \left(\frac{1}{c}\right) \iiint_{-\infty}^{+\infty} \frac{1}{|r - r'|} \begin{bmatrix} s_1(r', t') \\ s_2(r', t') \end{bmatrix}_{ret} dr'. \quad (70)$$

This is a well-known solution to the direct-space problem.

## 8.0 SUMMARY AND CONCLUSIONS

The Maxwell field equations have a natural matrix form (equation 9) where the matrix is not square. With the intuitive goal of making the matrix a square, and continuing the skew-Hermitian trend of the existing elements of the matrix, two columns are added with compensating zeros in the field vector; see equation 10. Some of the elements (see equations 12a through 14) remain unfixed. Interestingly, the particular choices (equations 13 or 14) for the elements lead to either the Lorentz-gauge or the Coulomb-gauge representations of the fields, respectively. Other choices for the elements will lead to other gauges.

The tactic of squaring matrix  $[M]$  has apparently not been done before, but has many benefits, as summarized next. Use of the square matrix operator  $[M]$  allows Maxwell's equations to be placed in a compact, covariant form (equation 16) (see appendix B for proof of covariance). Once in this form, use of the simple matrix operation of multiplication upon equation 16 derives, in turn, the landmark effects of electromagnetic theory: (a) the electromagnetic wave, and charge continuity equations, (equation 20); (b) the Lorentz-gauge / Coulomb-gauge definitions (equation 25) of the electromagnetic potentials; (c) the wave equations (equation 31) for the potentials; and (d) Poynting's theorem (equation 34) on energy conservation. These matrix-based derivations completely avoid the need for vector calculus and attendant special-purpose identities involving multiple curls and divergences. The necessary mathematical baggage is strongly reduced by the approach. Note, the extension of the matrix  $[M]$  into a square form, described above, is essential to the above electromagnetic derivations.

Alternatively, taking the four-dimensional Fourier transforms of equation 16 leads to (e) a Fourier representation (equation 45) of Maxwell's equations; (f) their inversion, (equation 48) for the fields directly in terms of the sources (but in Fourier-space); (g) corresponding inversion formulae (equations 58 and 59) in direct-space; and (h) corresponding results (see appendix A) to a through c preceding for the Coulomb gauge.

Results e through g are new in application to the full-fledged space-time problem. Analogous results have previously been found for the special case of time-harmonic fields (Mayes, 1967 and 1985).

Other standard results of electromagnetic theory follow as well from the matrix approach, but the need for brevity rules out their inclusion here. For example, a simple operation upon matrix equation 16 in the source-free case directly leads to the familiar requirement that the resulting electromagnetic waves should be transverse.

The matrix formulation involving the Lorentz gauge is covariant. All basic vector and matrix quantities are four-dimensional. However, the fact that there are eight scalar Maxwell equations causes a requirement that all four-dimensional quantities be "doubled up" into eight-dimensional quantities: Thus the two four-dimensional sources vectors  $[s_1], [s_2]$  are stacked vertically as one eight-dimensional vector, see equation 16; the four-dimensional matrices  $[M_1], [M_2]$  are blocked into a single square eight-by-eight matrix  $[M]$ ; etc. This also emphasizes that electromagnetic theory is basically eight-dimensional.

Because of the power of matrix operations, we were able to derive the key effects of electromagnetic theory without the need for the usual plethora of vector calculus identities that have become the standard in these derivations: Stokes' theorem, the Divergence theorem, Green's theorem, the formula for the curl of a curl, etc. Instead, the simple matrix operations of multiplication and inversion were used. We find these to be easier to understand and remember than the preceding vector calculus identities. Hence, we believe that electromagnetic theory is easier to learn (and is perhaps best taught) from the matrix viewpoint.

Aside from pedagogical aspects, we also found the matrix approach lends itself to convenient linear analysis via Fourier theory. Fourier theory is, of course, widely used and understood, because of its well-known applications in optical systems theory (Goodman, 1968), circuit theory, etc. Also, fast Fourier transform subroutines widely exist, so the approach is practical. Our application leads to a Fourier theory of electromagnetism, based on the use of *four-dimensional* Fourier transforms. Maxwell's equations, expressed this way in  $K$ -space, become a simple transfer theorem (equation 45) with a transfer function  $[K]$ . Very conveniently,  $[K]$  has a simple inverse, and this allows inversion (equation 48) to take place. Moreover, the inversion formulae equation 48 or equations 58 and 59 that result express the fields directly in terms of the sources. There is no need to calculate intermediary potentials first.

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## APPENDIX A COULOMB GAUGE

As previously indicated, the conditions of the Coulomb gauge are satisfied if the quantities  $\hat{m}_{44}$  and  $\hat{m}_{88}$  appearing in the Maxwell field equations 10 are set equal to zero. This is shown next. With  $\hat{m}_{44} = \hat{m}_{88} = 0$ , matrix  $[M]$  takes the form

$$[M] \equiv \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial \tau} & 0 & 0 & -\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 \\ 0 & -\frac{\partial}{\partial \tau} & 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 & -\frac{\partial}{\partial \tau} & 0 & 0 & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 & 0 & -\frac{\partial}{\partial \tau} & 0 & -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial z} \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \end{bmatrix}. \quad (A1)$$

Then equation 25 is explicitly the set of equations

$$E = -\nabla \phi^e - \left(\frac{1}{c}\right) \frac{\partial}{\partial t} A^e - \nabla \times A^m \quad (A2a)$$

$$H = -\nabla \phi^m - \left(\frac{1}{c}\right) \frac{\partial}{\partial t} A^m + \nabla \times A^e \quad (A2b)$$

and

$$\nabla \cdot A^e = 0 \quad (A3a)$$

$$\nabla \cdot A^m = 0 \quad (A3b)$$

Equations A2 define the electromagnetic fields in terms of the electromagnetic potentials, and equations A3 specifically define the Coulomb gauge (Jackson, 1962). This is what we set out to show.

Next, we multiply both sides of equation 16 by the complex conjugate of [M] given by equation A1. We obtain

$$\begin{bmatrix} D & O \\ O & D \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad (\text{A4})$$

where

$$\begin{bmatrix} D & O \\ O & D \end{bmatrix} = \begin{bmatrix} \square^2 & 0 & 0 & -\frac{\partial^2}{\partial x \partial \tau} & 0 & 0 & 0 & 0 \\ 0 & \square^2 & 0 & -\frac{\partial^2}{\partial y \partial \tau} & 0 & 0 & 0 & 0 \\ 0 & 0 & \square^2 & -\frac{\partial^2}{\partial z \partial \tau} & 0 & 0 & 0 & 0 \\ -\frac{\partial^2}{\partial x \partial \tau} & -\frac{\partial^2}{\partial y \partial \tau} & -\frac{\partial^2}{\partial z \partial \tau} & -\nabla^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \square^2 & 0 & 0 & -\frac{\partial^2}{\partial x \partial \tau} \\ 0 & 0 & 0 & 0 & 0 & \square^2 & 0 & -\frac{\partial^2}{\partial y \partial \tau} \\ 0 & 0 & 0 & 0 & 0 & 0 & \square^2 & -\frac{\partial^2}{\partial z \partial \tau} \\ 0 & 0 & 0 & 0 & -\frac{\partial^2}{\partial x \partial \tau} & -\frac{\partial^2}{\partial y \partial \tau} & -\frac{\partial^2}{\partial z \partial \tau} & -\nabla^2 \end{bmatrix} \quad (\text{A5})$$

(Compare with the corresponding Lorentz gauge equation 21.) Again  $\square^2$  is the D'Alembertian operator and  $\nabla^2$  is the Laplacian operator. The single matrix equation A4 is equivalent to equations 23 through 24 describing the electromagnetic field wave equations and the electric and magnetic charge continuity equations. These are the same as for the Lorentz gauge, as can be easily verified.



Finally, we substitute equation 25 back into Maxwell's equations (equation 16) with [M] given by equation A1. We obtain

$$\begin{bmatrix} D & O \\ O & D \end{bmatrix}^* \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \quad (\text{A6})$$

[D] is given in equation A5. This single matrix equation is equivalent to the following set of equations

$$\nabla^2 A^e - \left(\frac{1}{c^2}\right) \frac{\partial^2 A^e}{\partial t^2} = -\left(\frac{4\pi}{c}\right) J^e + \left(\frac{1}{c}\right) \nabla \left(\frac{\partial}{\partial t} \phi^e\right) \quad (\text{A7a})$$

$$\nabla^2 A^m - \left(\frac{1}{c^2}\right) \frac{\partial^2 A^m}{\partial t^2} = -\left(\frac{4\pi}{c}\right) J^m + \left(\frac{1}{c}\right) \nabla \left(\frac{\partial}{\partial t} \phi^m\right) \quad (\text{A7b})$$

and

$$\nabla^2 \phi^e = -(4\pi) \rho^e \quad (\text{A8a})$$

$$\nabla^2 \phi^m = -(4\pi) \rho^m. \quad (\text{A8b})$$

(Compare with the Lorentz gauge equations 32 through 33.) Equations (A7) represent the inhomogeneous wave equations for the electric and magnetic vector potentials, and equations A8 show the scalar electric and magnetic potentials each satisfy Poisson's equation. These are the known results for the Coulomb gauge.

## APPENDIX B

### COVARIANCE OF THE EIGHT-BY-EIGHT MATRIX REPRESENTATION OF MAXWELL'S EQUATIONS

In this appendix we show the eight-by-eight matrix representation (equation 16) of the Maxwell field equations is in covariant form. Consider two inertial right-handed Cartesian coordinate reference frames  $S$  and  $S'$  as shown in the figure below. These two frames share a common  $zz'$  axis. The frame  $S'$  is receding from the frame  $S$  with a constant speed  $v$  in the positive  $z$ -direction.

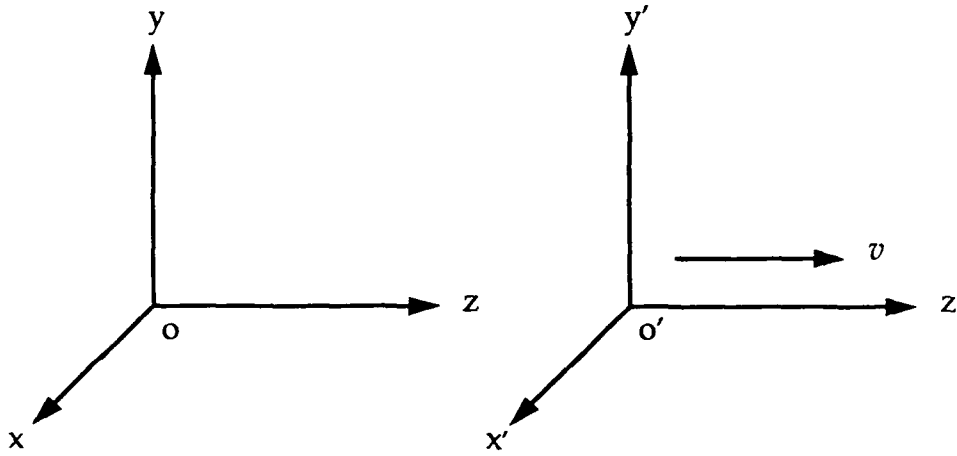


Figure B1. Two inertial reference frames  $S$  and  $S'$  in relative motion.

An observer in  $S$  reports space-time coordinates  $(x, y, z, t)$  for an event and an observer in  $S'$  reports  $(x', y', z', t')$  for the same event. For convenience we suppose that each observer chooses  $t = t' = 0$  to represent the instant that their origins coincide. We know that the space-time coordinates in  $S$  and  $S'$  are related through the Lorentz transformation (Jackson, 1962).

$$x'_\mu = \sum_{\nu=1}^4 L_{\mu\nu} x_\nu, \quad \mu = 1, 2, 3, 4 \quad (\text{B1})$$

with relativistic four-vectors

$$(x_1, x_2, x_3, x_4) = (x, y, z, \tau), \quad (x'_1, x'_2, x'_3, x'_4) = (x', y', z', \tau'). \quad (\text{B2})$$

Coefficients  $L_{\mu\nu}$  form the four-by-four Lorentz transformation matrix

$$\begin{bmatrix} L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix}. \quad (\text{B3})$$

The speed parameter,  $\beta$ , and the Lorentz factor,  $\gamma$ , are given by

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \quad (\text{B4})$$

The transformation properties of the electromagnetic field (Jackson, 1962) can be found from

$$F'_{\mu\nu} = \sum_{\lambda=1}^4 \sum_{\sigma=1}^4 L_{\mu\lambda} L_{\nu\sigma} F_{\lambda\sigma}, \quad \mu, \nu = 1, 2, 3, 4 \quad (\text{B5})$$

where  $F_{\lambda\sigma}$  are elements of the anti-symmetric field-strength tensor

$$\begin{bmatrix} F \end{bmatrix} = \begin{bmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}. \quad (\text{B6})$$

Now, by equation B6 expressed in the primed system, element  $F'_{41}$  is  $iE'_x$ ,  $F'_{42}$  is  $iE'_y$ , etc. Note, that elements  $iE'_x, iE'_y$ , etc., define vector  $[f'_1 \ f'_2]^T$ . Hence, transformation equation B5 may be used to link  $[f'_1 \ f'_2]^T$  to a linear superposition of unprimed elements  $F_{\lambda\sigma}$  which, again, by equation B6 are elements of  $[f_1 \ f_2]^T$ . These

are linear equations of the matrix form

$$\begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} = \begin{bmatrix} V & W \\ W & V \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} V & W \\ W & V \end{bmatrix}^* \begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} \quad (\text{B7})$$

where

$$\begin{bmatrix} V \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} W \end{bmatrix} = \begin{bmatrix} 0 & -i\gamma\beta & 0 & 0 \\ i\gamma\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (\text{B8})$$

It is of interest to note that the length of the eight-by-one electromagnetic field vector is invariant under the Lorentz transformation (equation B7). That is,

$$[f_1 \ f_2] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = [f'_1 \ f'_2] \begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix}. \quad (\text{B9})$$

For this reason, this vector could be referred to as an electromagnetic relativistic *eight-vector*. We are not aware that this concept has been used before.

Since  $[s_1]$  and  $[s_2]$  are relativistic four-vectors, then the eight-by-one source vector appearing in equation 16 transforms between the frames S and S' according to

$$\begin{bmatrix} s'_1 \\ s'_2 \end{bmatrix} = \begin{bmatrix} L & O \\ O & L \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \quad \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} L & O \\ O & L \end{bmatrix}^* \begin{bmatrix} s'_1 \\ s'_2 \end{bmatrix} \quad (\text{B10})$$

where "element" O is a four-by-four matrix of zeros defined by equation 21.

We are now in a position to show that equation 16 is in covariant form. Starting with the matrix representation of the Maxwell field equations in inertial frame S, namely

equation 16, we perform the following matrix multiplication operation

$$\begin{bmatrix} L & O \\ O & L \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} L & O \\ O & L \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \quad (\text{B11})$$

With the use of the left-hand equation in equation B10, we can rewrite equation B11 in the form

$$\begin{bmatrix} L & O \\ O & L \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} s'_1 \\ s'_2 \end{bmatrix}. \quad (\text{B12})$$

Next we replace the electromagnetic relativistic eight-vector in equation B12 by the right-hand equation in B7. This gives

$$\begin{bmatrix} L & O \\ O & L \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} V & W \\ W & V \end{bmatrix}^* \begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} s'_1 \\ s'_2 \end{bmatrix}. \quad (\text{B13})$$

With the use of the following partial differential operator transformation (Jackson, 1962)

$$\frac{\partial}{\partial x'_\mu} = \sum_{\nu=1}^4 L_{\mu\nu} \frac{\partial}{\partial x_\nu}, \quad \mu = 1, 2, 3, 4 \quad (\text{B14})$$

it can be shown that

$$\begin{bmatrix} M'_1 & M'_2 \\ M'_2 & M'_1 \end{bmatrix} = \begin{bmatrix} L & O \\ O & L \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} V & W \\ W & V \end{bmatrix}^*. \quad (\text{B15})$$

With the use of equation B15, equation B13 reduces to

$$\begin{bmatrix} M'_1 & M'_2 \\ M'_2 & M'_1 \end{bmatrix} \begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} = \left(\frac{4\pi}{c}\right) \begin{bmatrix} s'_1 \\ s'_2 \end{bmatrix} \quad (\text{B16})$$

which we recognize as the matrix representation (equation 16) of the Maxwell field equations in frame  $S'$ . This establishes the matrix representation (equation 16) of the Maxwell field equations is in covariant form.

## APPENDIX C

### INVERSION OF MATRIX [K]

Start with equation 41 and operate on both sides, on the left, by matrix operator  $[M]^*$ , giving

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} e^{+i(K \cdot R)} = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} e^{+i(K \cdot R)}. \quad (C1)$$

Consider the *right-hand side* of equation C1. Matrix  $[K]$  commutes with complex multiplier  $e^{+i(K \cdot R)}$ , to yield

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} e^{+i(K \cdot R)} = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* e^{+i(K \cdot R)} \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}. \quad (C2)$$

Next take the complex conjugate of equation 41, with  $K$  replaced by  $-K$ . The result is an identity

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* e^{+i(K \cdot R)} = \begin{bmatrix} -K_1 & -K_2 \\ -K_2 & -K_1 \end{bmatrix}^* e^{+i(K \cdot R)}. \quad (C3)$$

Using identity C3 in equation C2 gives

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} e^{+i(K \cdot R)} = - \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* e^{+i(K \cdot R)} \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}. \quad (C4)$$

Finally, the right-hand side becomes

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} e^{+i(K \cdot R)} = - \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} e^{+i(K \cdot R)} \quad (C5)$$

since, again, quantity  $e^{+i(K \cdot R)}$  commutes with matrix  $[K]$ .

Next, consider the *left-hand side* of equation C1. Matrix [M] follows the ordinary matrix rules of multiplication, so that it obeys the associative law. Then the left-hand side of equation C1 obeys

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} e^{+i(K \cdot R)} = \left( \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \right) e^{+i(K \cdot R)}. \quad (C6)$$

Now the product [M]\*[M] obeys, by equations 20 and 21

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} = \begin{bmatrix} D & O \\ O & D \end{bmatrix}. \quad (C7)$$

Hence, the left-hand side obeys,

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} e^{+i(K \cdot R)} = \begin{bmatrix} D & O \\ O & D \end{bmatrix} e^{+i(K \cdot R)}. \quad (C8)$$

By straightforward differentiation and the use of notation in equation 61b

$$\begin{bmatrix} D & O \\ O & D \end{bmatrix} e^{+i(K \cdot R)} = K^2 \begin{bmatrix} I & O \\ O & I \end{bmatrix} e^{+i(K \cdot R)} \quad (C9)$$

where  $K^2$  is the simple scalar given by equation 47. Then combining equations C8 and C9, the left-hand side is

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} e^{+i(K \cdot R)} = K^2 \begin{bmatrix} I & O \\ O & I \end{bmatrix} e^{+i(K \cdot R)}. \quad (C10)$$

Then since the left-hand side and right-hand sides of equation C1 are equal, by equations C5 and C10

$$-\begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} = K^2 \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \quad (\text{C11})$$

or

$$\left(-\frac{1}{K^2}\right) \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}. \quad (\text{C12})$$

By definition of the inverse, this implies that

$$\begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^{-1} = \left(-\frac{1}{K^2}\right) \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^*. \quad (\text{C13})$$



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Code 75	J. E. Griffin	(1)
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Code 7601	J. R. Zeidler	(1)
Code 7601	K. Bromley	(1)
Code 7602	S. Hoff	(1)
Code 761	H. M. Berry	(1)
Code 761	R. P. Bocker	(60)
Code 761	G. W. Byram	(1)
Code 761	S. I. Chou	(1)
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Code 761	J. P. Loughlin	(1)
Code 765	E. Munn	(1)
Code 80	K. D. Regan	(1)
Code 805	J. W. Rockway	(1)
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Code 808	G. W. Beaghtler	(1)
Code 82	R. J. Kochanski	(1)
Code 824	J. C. Logan	(1)
Code 832	P. M. Hansen	(1)
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Code 844	B. James	(1)

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